Excitation of Surface Waves by a Finite Seismic Source: The Case of Gravity Waves in the Liquid Layer-Elastic Half-Space

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ABSTRACT

The excitation of gravity waves by both point and finite dimensioned line sources was considered in the flat elastic-gravitational model. The excitation functions were analyzed in terms of normal mode formalism in a far-field approximation of the surface wave.

A line source was modeled by a set of point sources, uniformly distributed along a fault and moving along it with a constant velocity. A detailed analysis of the double-couple mechanism shows that a line source with a vertical dip-slip motion generates more intensive gravity waves than one with a strike-slip motion. The calculation of theoretical marigrams for gravity waves demonstrates that the amplitude of a wave generated by a line source is larger than that for a point source of the same seismic moment.

A numerical experiment reveals that the rigidity of the rock and the source duration both exert a strong influence on the amplitude of the gravity waves generated by an underwater earthquake. A slow rupture with velocity 1-2 km/s occurring in low rigidity rock generates a more intensive gravity wave. It is suggested that the use of the more realistic finite dimensioned line source is appropriate to model “tsunami-earthquake” events.

(Key words: Gravity wave, Double-couple mechanism, Point source, Line source, Source-time function, Excitation function)

1. INTRODUCTION

In this paper we present an analytical model for investigation of the excitation and propagation process of special type of a surface waves the gravity wave. We are focusing on it for two main reasons:

First, the proposed model can be appropriate for the study of the excitation of abnormal

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large seismic sea waves (long-period gravity waves) from tsunami-earthquakes (such as 1963, 1975 Kurile Islands, 1992 Nicaragua; 1999 Papua New Guinea). Up to now the mechanism some of them is not clearly understood. A number of hypothesis have been proposed to explain anomalously large excitation of long period gravity waves, among them the source rupture times of unusually long duration (Kanamori 1972), non-double couple sources such as submarine landslides (Hasegawa and Kanamori 1987), an underestimation of the actual seismic moment from surface waves due to shallow source depth and fault angle (Ward 1982), and shallow faulting within sediments accompanied by a change in fault orientation during the rupture (Fukao 1979; Okal 1988).

The second reason of our choice of the gravity waves as an object of investigation is related to the study of complicated behavior of soils during the earthquake. With analytical recipes, described here, the mechanism of gravity wave excitation and their propagation in soils can be better understood. The 1985 Mexico earthquake is an example of gravity waves existence in soft clays and their responsibility for a considerable damage in the Mexico City (Lomnitz 1990).

In the present work, applying the normal mode formalism to the flat ocean-Earth model, we attempted to solve the one-dimensional problem in the case of a gravity wave caused by both point and line sources. The finite dimensioned source in our model not only brings more realism into the description of the gravity wave generation process but also is better suited for a study of the influence of the elastic parameters of the ocean bottom on the excitation of gravity waves (Bilek and Lay 1999). The model with low-rigidity half-space, which we used for computations, gives an opportunity to study in detail the influence of the sedimentary structure on the gravity wave amplitude. It is important, as mentioned above, in the studies of the tsunami-earthquake mechanism. Observational material collected at present suggest that a «very anomalous» (Kanamori and Kikuchi 1993) tsunami earthquake might occur at an accreting margin with large amounts of sediment.

Previously, Ward (1980, 1981, 1982) has undertaken similar studies and described the long period gravity wave (tsunami) generation in a spherically-symmetric, self-gravitating and elastic ocean-Earth model in terms of the normal mode theory. Ward modeled a subma

riner earthquake by both point and finite dimensioned line sources. He has formulated the equations of motion in a manner originally applied to the Earth’s free oscillations, and made use of a very general result to obtain tsunami mode excitation. From the analysis of the directivity of tsunami radiation and energy estimations, he has concluded that tsunami radiation patterns from finite sources are more uniform than those from a point source, and therefore, use of finite fault models in tsunami forecasting is desirable. In terms of the Earth’s free oscillations, McCowan (1976) has derived expressions for seismic energy radiated from point and line sources and for the power spectra corresponding to various fault length, rupture velocity and source duration. Okal’s (1988) study was also mostly based on the Ward formalism. Using a line model of the earthquake source, he has carried out a more sophisticated analysis of the influence of focal parameters and the directivity effects on tsunami generation.

Unlike Ward, we use a flat ocean-Earth model in this study and carry out calculations for the whole frequency range rather than for low frequencies only. Since gravity wave is an interference surface wave, it is reasonable to describe it using an approach based on the expan-
sion of the solution into eigenfunctions of the boundary problem (Levshin 1973). One advantage of this method is its applicability both to a laterally homogeneous medium and to a medium with weak lateral inhomogeneity (Keilis-Borok 1989).

It is unfortunate that, although the problem of generation and propagation of gravity waves requires an in-depth theoretical investigation, most studies in this direction are largely concentrated on the numerical modeling of the propagation of tsunamis excited by particular events, for instance, Nicaragua 1992 (Imamura and Shuto 1993), Peru 1996 (Heinrich et al. 1998), Papua New Guinea 1998 (Tanioka 1999) and others. In these studies the real bathymetry are used for numerical computations of waveform. In all cases, mentioned above, good agreement between the observed and the synthetics tsunami was obtained.

The present paper, while indeed based on the newest findings of Bilek and Lay (1999), is an attempt to change somewhat at the standard point of view on the problem of excitation of gravity waves by seismic sources, thereby filling a long-standing gap in this domain.

2. FORMULATION OF THE PROBLEM

A compressible liquid layer of uniform thickness $H$ resting on the elastic half-space was used to model the ocean-lithosphere system. The equations of motion are written in the linear theory approximation (Pod'yapol'sky 1968, 1970):

$$c_f^2 \nabla \text{div} \mathbf{u}_1 - \mathbf{g} \cdot \nabla \text{div} \mathbf{u}_1 = \frac{\partial^2 \mathbf{u}_1}{\partial z^2} \quad \text{at } 0 < z < H ; \quad \text{and}$$

$$ (\lambda + 2\mu) \nabla \text{div} \mathbf{u}_2 - \mu \nabla \times (\nabla \times \mathbf{u}_2) = \rho \frac{\partial^2 \mathbf{u}_2}{\partial z^2} \quad \text{at } z > H ,$$

where $c_f$ is the velocity of the acoustic wave in the liquid layer; $\lambda$ and $\mu$ are the Lame parameters in the elastic half-space; $\rho$ is the density of the elastic medium; $\mathbf{u}_i = (U_i, 0, W_i)$ is the displacement vector ($i=1$ liquid layer, $i=2$ elastic half-space); $\mathbf{g}$ is gravity; $\omega$ is the angular frequency; and $z$ is the depth from the free surface (positive downwards). Expressions for the horizontal and vertical components of the displacement are given in the Appendix of this paper.

The boundary conditions for the solution of these equations of motion are as follows: (1) pressure vanishes at the surface of the liquid layer; (2) the normal components of the displacement and strain at $z=H$ are continuous; (3) since fluid cannot sustain shearing stress, continuity of stress at the interface requires that the tangential component of stress be zero; and (4) at $z \to \infty$, displacement $u_z$ vanishes.

We considered the special case of plane waves propagating in the horizontal direction of increasing $x$, with frequency $\omega$ and the amplitude diminishing exponentially in the $z$ direction in the half space:

$$\mathbf{u}_i(x, z, \omega, t) = \mathbf{v}_i(z, \omega) \exp\left[i\omega(t - x/c)\right] .$$

Only $z$-derivatives remain in the equations of motion and boundary conditions, and such a one-dimensional problem is relatively simple to solve (Aki and Richards 1980).
3. MODAL CONTRIBUTION TO THE SOLUTION

To describe gravity wave displacement relative to the source-receiver location, we employed two sets of coordinates with origins at the earthquake: a Cartesian set \((x, y, z)\) and a cylindrical one \((z, r, \phi)\) in which angle \(\phi\) is measured clockwise from the x-axis. A force applied at the source was projected onto the Cartesian unit vectors \(e_1, e_2, e_3\), and displacements in the receiver point were projected onto unit vectors of the cylindrical system (Keilis-Borok 1989).

Applying the far-field approximation in the description of the surface waves generated by a point seismic source, it is possible to use relatively simple expressions for the surface wave displacement in the frequency domain. This can be obtained by differentiating the second rank tensor Green’s function. Generally, for Rayleigh and Love surface waves, the displacement is represented as the sum of the contributions from individual surface wave modes (Aki and Richards 1980). In the case of gravity waves, there is no summation over the higher modes. Finally, the far-field displacement of a gravity wave generated by a point source at the depth \(h\) in the solid Earth can be represented by:

\[
u(r, \phi, \omega, t) = \frac{\exp(-i\pi/4) \exp(i\omega t - r/c)}{\sqrt{8\pi}} \frac{U(z, \omega)Q(h, \phi, \omega)}{\sqrt{c u I_0}}.\]

In this formula, \(U(z, \omega) = U(z, \omega)e_\phi + W(z, \omega)e_\rho Q(h, \phi, \omega) = m_{rs}(\omega)B_{rs}(h, \phi, \omega)\) is the excitation function, \(m_{rs}(\omega)\) is the spectrum of the seismic moment tensor, \(B_{rs}(h, \phi, \omega)\) is a tensor depending on the axis orientation of the source which can be expressed via the eigenfunctions \(U(z, \omega)\) and \(W(z, \omega)\),

\[I_0 = \int_0^\infty \rho [U^2(z, \omega) + W^2(z, \omega)] \, dz\]

is the energy integral; \(c\) and \(u\) are the phase and group velocities of the gravity wave, respectively; \(r\) is the horizontal distance from the source; \(h\) is the source depth; \(\phi\) is the azimuth of an observation point.

In formula (4), the first and second terms describe respectively the source phase and the phase shift during propagation and the effect of geometrical divergence of the energy flow on the propagation of the gravity wave. The third term is depends on the depth of the receiver, and the fourth term depends on depth, focal mechanism and the radiation spectrum in the half-space.

4. GRAVITY WAVES GENERATED BY DOUBLE-COUPLE SOURCES

The simplest possible seismic source is a single force applied to a point. This mechanism does not produce faulting, but rather moves in one direction. Such a mechanism represents longitudinal motion and is often associated with landslides. However, Okal (1990) used single-force source representation to describe surface and tsunami waves excitation processes. He showed that an average single force, whose orientation and exact depth are unknown, should excite any seismic wave proportionally to wavelength, relative to a similarly average double-
couple with the same source-time function.

A force couple is the minimum combination of single forces for the representation of an earthquake. In this source mechanism, two forces pointing in opposite directions are separated by a small distance. A single couple possesses a moment at about its epicenter and thus exerts a net torque on the Earth. When a second force couple is applied perpendicular to the first one, it exerts an equal and opposite torque and hence, no net torque exists. The radiation pattern for a double couple may be obtained by differentiation with respect to the source coordinate. That is, the double-couple radiation pattern is the derivative of a single force along the T-axis* minus the derivative of a single force along the P-axis**.

There are two commonly used descriptions of point sources. The first is in terms of an angular description of the nodal planes in P radiation from a pure slip motion on a fault (Kennett 1988). The second, and one of growing importance, is the description of a source using six independent components of the moment tensor, all having a common dependence on time. The tensor model is capable of representing curved as well as planar faults and has become the canonical point-source model of an earthquake. The linear dependence of the response upon the elements of $M$ makes the moment tensor representation particularly advantageous in source-mechanism studies. We will use in our modeling the second way of the description of point source.

In the present study, we analyzed two basic seismic geometries: a strike-slip earthquake on a vertical fault and a dip-slip earthquake on a vertical fault. These represent the lower and upper limits of efficiency in gravity wave generation. Any other double-couple mechanism can be obtained as a linear superposition of the above two mechanisms.

4.1 Strike-slip Fault

A single couple consists of two forces oriented in the x-direction and offset in the y-direction. The force couple, $m_{yx}$, is the torque, equal to the distance between the forces times the force. A double-couple system includes an additional force couple, $m_{xy}$ (forces oriented in the y-direction and offset in the x-direction). Both $m_{yx}$ and $m_{xy}$ act in opposite directions and therefore the net torque is zero. The double-couple source then becomes the sum of the two force couples. In general, the fault orientation is not the plane of the coordinate axes, and the other elements of the matrix, $m_{xz}$, $m_{yz}$, $m_{zx}$, $m_{zy}$, $m_{zz}$ are non-zero. Thus, the moment tensor is a 3-by-3 symmetric matrix.

For any point source, the normalized excitation function can be written as (Keilis-Borok 1989)

$$Q(h,\omega,\varphi) = M_0 g(h,\omega,\varphi) F(\omega),$$  (5)

where $M_0$ is the scalar seismic moment; $I_0$ is the energy integral. Function $q(h,\omega,\varphi)$ depends on the material properties, and the source movement parameters are expressed as follows:

*T-tension axis represents the direction of minimum compressive stress.

**P-compression axis represents the direction of maximum compressive stress.
\[ q(h, \omega, \phi) = -\xi U_z(h, \omega)\sin 2\phi \sqrt{c u I_0}. \]  

\( \xi = \omega / c \) is the wave number; \( U_z \) is the horizontal component of displacement in the elastic half-space.

The spectrum of the seismic moment \( F(\omega) \) in equation (5) can be evaluated from the seismic moment growing from zero to a constant value \( M_0 \) over a certain time interval called rise time \( \tau_0 \). Assuming that the source time function has the form \( 1 - \exp(-t/\tau_0) \) at \( t > 0 \) and vanishes at \( t < 0 \), we obtain:

\[ F(\omega) = \frac{1}{i\omega(i\omega\tau_0 + 1)}. \]  

The specific form of the time function has a minor effect on the final result because any other function that increases in the time interval \( \tau_0 \) has the same spectral features such that \( F(\omega) \sim \omega^{-1} \) at low frequencies and \( F(\omega) \sim \omega^{-3} \), at high frequencies.

### 4.2 Dip-slip Fault

The orientation of the fault plane is assumed to be the same as in the former case, but the slip occurs along the z axis. In this case, the non-zero components of the moment tensor are:

\[ m_{xz} = m_{zx} = M_0 F(\omega). \]

Similar to the previous case, the function \( q(h, \omega, \phi) \) for a vertical normal fault is expressed as:

\[ q(h, \omega, \phi) = \frac{\text{ics}\phi}{\sqrt{c u I_0}} \left[ -\xi W_z(h, \omega) + \left. \frac{d U_z(z, \omega)}{dz} \right|_{z=b} \right]. \]

\( U_z \) and \( W_z \) are respectively the horizontal and vertical components of displacement in the half-space.

### 5. GRAVITY WAVES GENERATED BY FINITE DIMENSIONED LINE SOURCES

With numerical modeling of surface wave generation, it is often possible to ignore the finite size of the seismic source and use a point-source approximation. This is justified when the characteristic periods observed in the seismic record are long compared with the duration of the source and when the wavelengths are much longer than the source dimensions. Basically line source can be constructed in two different ways. One is to consider a distribution of point sources over finite region acting simultaneously. Alternatively, representation can be a single point source tracing out a moving pattern over a finite volume of material (Aki and Richards 1980; Kennett 1988).
Following that we bring more realism into the description of the process of gravity wave generation, and extend the limits of the investigations using a distributed source. We considered the case where the fault length greatly exceeds its width i.e., one that is essentially one-dimensional.

Taking into account the relation between moment tensor (M) and the moment density tensor (m) in the form:

\[ M_{pq} = \int_{L} m_{pq} dL, \tag{9} \]

the normalized excitation function in equation (5) in the case of a finite dimensioned line source becomes:

\[ \hat{Q}(h, \omega, \phi) = \frac{1}{\sqrt{cu L_0}} \int_{0}^{L_0} m(\omega, L) \exp \left[ -i\omega L \left( \frac{1}{V} - \frac{\cos \alpha}{c} \right) \right] Q(h^1, \omega, \phi) dL, \tag{10} \]

where \( h^1 = h_0 - L \sin \alpha \) is the depth of the moving point at the fault; \( m \) is the seismic moment density \( \int_{0}^{L_0} m(\omega, L) dL = M_0(\omega) \); \( L_0 \) is the fault length; \( V \) is the velocity of the rupture propagation; \( c \) is the phase velocity; \( \alpha \) is the azimuth of observation relative to the direction of the rupture; and \( L \left( \frac{1}{V} - \frac{\cos \alpha}{c} \right) \) is the rupture time.

The rupture time is the second factor (besides rise time) which characterizes the source-time function. The rupture time, the time required for the rupture to propagate across the entire fault, is determined by rupture velocity and by the orientation of the station relative to the fault. Evidently, in the case of gravity wave excitation by a point source, the source-time function depends only on the rise time. However, in the case of a line source, the source-time function depends on both the rise time and rupture time.

Since the finite width is not taken into account in the considered case of a distributed source, the spectrum attenuates as \( \omega^{-2} \) at high frequencies.

In light of the fact that the earthquake rupture velocity \( V \) is typically about 80% ± 10% of the shear velocity based on the standard definitions of seismic moment and empirical observations (Bilek and Lay 1999), formula (5) can be rewritten as:

\[ \frac{Q(h, \omega, \phi)}{\sqrt{cu L_0}} = \int_{0}^{L_0} m(\omega, L) \exp \left[ -i\omega L \left( \frac{1}{0.8 \sqrt{\mu / \rho}} - \frac{\cos \alpha}{c} \right) \right] \frac{Q(h^1, \omega, \phi)}{\sqrt{cu L_0}} dL, \tag{11} \]

where \( \mu \) and \( \rho \) are the rigidity and rock density, respectively.
6. RESULTS OF NUMERICAL MODELING

Here, a numerical experiment was carried out for the system consisting of a liquid layer with density $\rho = 1 \text{g/cm}^3$, a velocity of acoustic wave $c_f = 1.45 \text{km/s}$ a layer thickness of 4 km, and half-space with the elastic parameters given in Table 1. Using such parameters for numerical modeling we follow the global model of the Earth’s crust proposed by Mooney (1998), which is based on seismic reflection data. In all model computations, the source depth was 45 km corresponding to the midpoint of a line source (fault). Gravity wave synthetics were calculated in the azimuth ($\varphi$) of the maximum of the radiation pattern.

Figure 1(a) shows the behavior of the normalized excitation function (formula (11)) for line sources with two different types of motion. A line source with the dip-slip motion obviously generates stronger gravity waves than one with a strike-slip motion, especially in the high-frequency range. This agrees with the results obtained by Ward (1982).

In the whole frequency range, the excitation function of the line source (Fig. 1a) has much greater power compared with that of the point source (Fig. 1b). This feature also holds in theoretical marigrams. The amplitude of the gravity wave generated by a line source (Figs. 2a and 3a) with either type of motion is more than one order of magnitude higher than the amplitude of the wave generated by a point-source (Figs. 2b and 3b). The time domain waveforms of the gravity wave due to the point moment tensor were obtained by applying the inverse Fourier transformation.

Using an elastic medium with parameters corresponding to hard rock (model 1 in Table 1), hard sediments (model 2) and soft sediments (model 3), we investigated the influence of rock rigidity and source duration on the amplitude of the gravity waves. The far field synthetics of the gravity waves (vertical component) calculated at the 1000 km epicentral distance for various medium models are shown in Fig. 4 for a line source with dip-slip motion and in Fig. 5 for a line source with strike-slip motion. Evidently, the slow earthquake rupture occurring in the low-rigidity media generates the strongest gravity waves. The obtained results are in good agreement with recent observations of Bilek and Lay (1999) who carried out experiment for the low-frequency part of a gravity wave -tsunami wave.

7. CONCLUSIONS

In this paper we have analyzed the influence on the gravity wave excitation of such parameters as finiteness of seismic source, mechanism, and its location in the structures with

![Table 1]

<table>
<thead>
<tr>
<th>Models</th>
<th>Rock density [g/cm$^3$]</th>
<th>Wave velocities [km/s]</th>
<th>Source duration [s]</th>
<th>Rupture velocity [km/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>N1</td>
<td>3.1</td>
<td>7.15, 4.1</td>
<td>32</td>
<td>3.28</td>
</tr>
<tr>
<td>N2</td>
<td>2.5</td>
<td>4.3, 2.5</td>
<td>62</td>
<td>2.0</td>
</tr>
<tr>
<td>N3</td>
<td>1.5</td>
<td>2.0, 1.15</td>
<td>83</td>
<td>0.92</td>
</tr>
</tbody>
</table>
Fig. 1. Dependence of $q(h, \omega, \phi)$ on period for two kinds of seismic sources in the model N1: a) distributed source with dip-slip motion (solid line; $\phi=0^\circ$); and with strike-slip motion (dashed line; $\phi=45^\circ$); b) dip-slip point-source ($\phi=0^\circ$).

Fig. 2. Comparison of the theoretical dip-slip marigrams of gravity waves from (a) distributed and (b) point dip-slip seismic sources in the model N1.

different rigidity. The gravity waves have been studied in terms of the far-field approximation of normal mode formalism. We extended the frequency range of investigation and completed the results Bilek and Lay not limiting only by the low-frequency component- tsunami. The
Fig. 3. Comparison of the theoretical strike-slip marigrams of gravity waves from (a) distributed and (b) point strike-slip seismic sources in the model N1.

Fig. 4. Comparison of the theoretical dip-slip marigrams of gravity waves from distributed seismic source for different medium models: (a) model N3; (b) model N2; (c) model N1. At the right figure wave amplitudes in cm are shown.
line source was modeled by a set of point sources, uniformly distributed along the fault and moving along it with a constant velocity. To study the influence of rock rigidity on gravity wave excitation, we placed the seismic source in media with different rigidity parameter. Numerical modeling yielded the following results:

- The amplitude of a gravity wave generated by a line source is stronger than that by a point source.
- The line source with a dip-slip motion is two times more effective compared to that with a strike-slip motion.
- Rupture propagating into a sedimentary structure plays an important role in gravity wave excitation: within a low rigidity medium a line source generates four times stronger gravity wave in the whole frequency range than in typical crustal rocks.
- Source parameters, such as the velocity of rupture propagation and source duration, as well as the depth of the source and rock rigidity are essential in the consideration of gravity wave generation.

Fig. 5. Comparison of the theoretical strike-slip marigrams of gravity waves from distributed seismic source for different medium models: (a) model N3; (b) model N2; (c) model N1. At the right figure wave amplitudes in cm are shown.
We conclude here that in spite of that the numerical simulation now is used as one of the most effective means in the practical design of tsunami defense works, analytical modeling is still necessary to have good understanding of the parameters bearing the primordial influence on tsunami excitation.

Further on, the analytical model developed here will be used to investigate the excitation and propagation of gravity and Rayleigh waves in multilayered media both with and without a superficial liquid layer. The results of the present study combined with those of Ward (1982) can also be useful for the calculations of energy and the efficiency of gravity and Rayleigh waves for seismic sources with different time histories.

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APPENDIX

The expressions for wave amplitudes as a function of depth in the liquid layer are as follows:

\[ U_z(z, \omega) = \frac{ic^2}{\omega \xi} \left[ -B e^{-\eta_1 z/c_f} + C e^{\eta_1 z/c_f} \right] \text{ and } \]

\[ W_z(z, \omega) = -\frac{c_f}{\omega^2} \left[ -\eta_1 B e^{-\eta_1 z/c_f} + \eta_2 C e^{\eta_1 z/c_f} \right] \]

where:

\[ \eta_1 = -\omega \gamma - \frac{g}{2c_f}; \]

\[ \eta_2 = \omega \gamma - \frac{g}{2cf}; \text{ and } \]

\[ \gamma = \frac{c^2}{c} - 1 + \frac{g^2}{4c^2 \omega^2}. \]

In the half-space, these are:

\[ U_z(z, \omega) = \frac{i\alpha^2}{\omega \xi} D e^{-\omega \alpha(z - H)/a} - \frac{ib\beta}{\omega} F e^{(-\omega \beta(z - H)/b)}; \text{ and } \]

\[ W_z(z, \omega) = -\frac{c_f}{\omega^2} \left[ -\eta_1 B e^{-\eta_1 z/c_f} + \eta_2 C e^{\eta_1 z/c_f} \right] \]
\[ W \zeta(z, \omega) = \frac{a\alpha}{\omega} \left(1 + \frac{g}{2\omega\alpha} D \exp(-\omega\alpha + \frac{g}{2a})(z - H)/a\right) - \frac{b^2}{\alpha c} F \exp(-\omega\beta(z - H)/b), \]

where \( \alpha^2 = a^2 / c^2 - 1 \) and \( \beta^2 = b^2 / c^2 - 1 \).